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Consumption Smoothing and Discounting in Infinite-Horizon, Discrete-Choice Problems

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Abstract. Suppose the consumption space is discrete. Our first contribution is a technical result showing that any continuous utility function of any stationary preference relation over infinite consumption streams has convex range, provided that the agent is sufficiently patient. Putting the result to use, we consider a model of endogenous discounting (a generalization of the standard model with geometric discounting) and show the uniqueness of the consumption-dependent discount factor as well as the cardinal uniqueness of utility. Comparative statics are then provided to substantiate the uniqueness. For instance, we show that, as in the more familiar case of an infinitely divisible good, the cardinal uniqueness of utility captures an agent's desire to smooth consumption over time.

Keywords: dynamic choice • endogenous discounting • discrete outcomes • expansions of a real number

1. Introduction

Consider the standard model of intertemporal choice in which an agent's preferences over consumption sequences can be represented by the utility function:

$$U(c_0, c_1, c_2, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t). \tag{1.1}$$

As with any representation of behavior, one wants to know whether the parameters of the representation are unique, which invariably suggests that they have a well-defined meaning in terms of the underlying preferences of the agent. In the context of (1.1), a classical result by Koopmans [14] shows that if the space C of possible consumption outcomes is connected, then the discount factor $\beta \in (0,1)$ is unique, and the instantaneous utility function $u : C \to \mathbb{R}$ is unique up to positive affine transformations. Moreover, as is well understood, β can be interpreted as a measure of patience, whereas the curvature of u captures the agent's desire to smooth consumption over time.

What happens when consumption is indivisible and, in particular, when C is a finite set? Are the parameters of familiar representations such as (1.1) still unique? If so, can one flesh out the behavioral meaning behind them? Our analysis of these questions begins with a technical lemma showing that *any* continuous utility function $U : C^\infty \to \mathbb{R}$ of *any* stationary preference $\succeq$ on a space C^∞ of consumption sequences has convex range, provided that the agent is sufficiently patient. The lemma generalizes a well-known result in number theory, due to Rényi [21], which shows that every number $k \in [0,1]$ is the discounted average $(1-\beta)\sum_t \beta^t u_t$ of some sequence $(u_0, u_1, \ldots) \in \{0,1\}^\infty$, provided that $\beta \geq \frac{1}{2}$. Because any discounted average is the continuous utility function of a stationary and *time-separable* preference on C^∞, we see that our lemma generalizes Rényi [21] by dropping the assumption of time-separable utility.

A practical implication of this generalization is that we can consider stationary but nonseparable models such as the one introduced by Uzawa [24], in which utility takes the form

$$U(c_0, c_1, \ldots) = u(c_0) + b(c_0)u(c_1) + b(c_0)b(c_1)u(c_2) + \ldots. \tag{1.2}$$

Note that if the function $b : C \to (0,1)$ is constant, then (1.2) reduces to (1.1), with b turning into the familiar discount factor β. One can thus view (1.2) as a generalization of the standard model in which the rate of time preference may vary endogenously with consumption. Despite this endogeneity, we can still show that, if the agent is sufficiently patient, the function $b : C \to (0,1)$ is unique and the function $c \mapsto U(c, c, \ldots) \equiv u(c)(1 - b(c))^{-1}$

is unique up to positive affine transformations. These conclusions match those of Bommier et al. [4] obtained under the assumption of a connected outcome space C and, when discounting is exogenous, those of Koopmans [14].

To gain intuition for this uniqueness result and the use of our lemma, it is helpful to rewrite the Uzawa model recursively as follows:

$$U(c_0, c_1, c_2, \ldots) = (1 - b(c_0))U(c_0, c_0, c_0, \ldots) + b(c_0)U(c_1, c_2, c_3, \ldots). \tag{1.3}$$

We see that the utility of any stream $(c_0, c_1, c_2, \ldots)$ is a convex combination of $U(c_0, c_0, \ldots)$ and $U(c_1, c_2, \ldots)$. But from our lemma, the latter utility is drawn from a connected set. Thus, the recursive structure of stationary preferences, together with our lemma, means that we can overcome the discreteness of the outcome space C by expressing intertemporal trade-offs in terms of the connected space of continuation utilities.

Our final contribution is to clarify the behavioral implications of our uniqueness result. Section 3.1 shows that, as in the case of divisible consumption, the cardinal uniqueness of utility reflects an agent's desire to smooth consumption over time. Turning to patience, Section 3.2, presents behavior that is necessary for one agent to be uniformly more patient than another, by which we mean that, for the first agent, $b(c)$ is greater for each $c \in C$. When discounting is exogenous, as in (1.1), we can go a step further and use observed behavior to measure an agent's discount factor β up to any level of precision.[1]

2. A Workhorse Lemma

Presently, we introduce the formal setting, the important class of stationary preferences, and our main result.

Let C be a finite space of consumption outcomes, with at least two elements, and C^∞ the space of all consumption sequences $(c_0, c_1, \ldots)$, where $c_t \in C$ denotes the outcome consumed in period $t \in \{0, 1, 2, \ldots\}$. We endow C with the discrete topology and C^∞ with the respective product topology. The latter renders C^∞ a compact, second-countable space. A *preference relation* on C^∞ is a complete and transitive binary relation $\succeq$.[2] To simplify the exposition, we assume, without further mention, that if $c \neq c'$, then the constant sequences $(c, c, \ldots), (c', c', \ldots)$ are not indifferent. This assumption makes sense given the finiteness of C and is largely without loss of generality (w.l.o.g.). Our main results go through without it. At the end of Section 4, we discuss some results that do not.[3]

We study preferences $\succeq$ that satisfy three additional restrictions. The first is continuity. Among its many benefits, this assumption is important in applications as it ensures the existence of a preference-maximizing element in any compact set $K \subset C^\infty$. To state the assumption, recall that given a preference relation $\succeq$ on C^∞, the *upper contour set* of a sequence $(c_0, c_1, \ldots)$ is the set of all sequences $(c'_0, c'_1, \ldots)$ such that $(c'_0, c'_1, \ldots) \succeq (c_0, c_1, \ldots)$. Reversing the latter ranking gives *the lower contour set*.

Assumption 1 (Continuity). *The upper and lower contour sets of $\succeq$ are closed in the product topology.*

The next assumption is Koopmans's notion of stationarity, and the most substantive assumption we impose. It says that the passage of time has no effect on preferences.

Assumption 2 (Stationarity). *For all $z \in C$ and $(c_0, c_1, \ldots), (c'_0, c'_1, \ldots) \in C^\infty$,*

$$(c_0, c_1, \ldots) \succeq (c'_0, c'_1, \ldots) \quad \text{if and only if} \quad (z, c_0, c_1, \ldots) \succeq (z, c'_0, c'_1, \ldots).$$

At this stage, it may be beneficial to make a short detour and recall a result of Koopmans [13], which shows that stationary preferences admit a particularly useful recursive representation. In fact, many readers may recognize the class of stationary preferences from this representation, which is how the preferences are typically specified in practice. See, for instance, Lucas and Stokey [16] and the surveys by Epstein and Hynes [9] and Backus et al. [3]. First, we recall that a function $U : C^\infty \to \mathbb{R}$ is said to be a utility function for a preference relation $\succeq$ on C^∞ if

$$(c_0, c_1, \ldots) \succeq (c'_0, c'_1, \ldots) \Leftrightarrow U(c_0, c_1, \ldots) \succeq U(c'_0, c'_1, \ldots). \tag{2.1}$$

A utility function of $\succeq$ will also be called a *representation* of $\succeq$. We note that utility functions are not unique; any strictly increasing transformation of U gives another utility function of the same preference relation.

Lemma 1. *A preference relation $\succeq$ on C^∞ is continuous and stationary if and only if it has a continuous utility function $U : C^\infty \to \mathbb{R}$ and every such function takes the form*

$$U(c_0, c_1, c_2, \ldots) = \phi(c_0, U(c_1, c_2, \ldots)), \tag{2.2}$$

for some $\phi : C \times U(C^\infty) \to \mathbb{R}$, continuous and strictly increasing in its second argument.

Proof. The proof is essentially that of Koopmans [13]. We will sketch a proof for the sake of completeness and to make clear that some existing differences in the setup and assumptions are essentially immaterial. The existence of a continuous utility function $U: C^\infty \to \mathbb{R}$ follows from Debreu [6], given that C^∞ is second countable. Take such a function. For every $c \in C$ and $k \in U(C^\infty)$, let $(c_0, c_1, \ldots) \in C^\infty$ be such that $U(c_0, c_1, \ldots) = k$, and let $\phi(c, k) = U(c, c_0, c_1, c_2, \ldots)$. By stationarity, the function ϕ is well defined and strictly increasing in its second argument. It is continuous because U is continuous. The recursive representation in (2.2) follows by construction. □

We turn to our final assumption. It has been known since Diamond [7] that any continuous preference $\succeq$ on a domain of infinite streams must exhibit some degree of impatience. For example, the discounted sum in (1.1) will blow up unless $\beta < 1$. Our next assumption serves as a lower bound on the degree of impatience. To state it, we note that if $\succeq$ is continuous and stationary, then the best and worst consumption sequences can be chosen to be constant,[4] that is, there are outcomes $c_*, c^* \in C$ such that

$$(c^*, c^*, \ldots) \succeq (c_0, c_1, \ldots) \succeq (c_*, c_*, \ldots) \quad \forall (c_0, c_1, \ldots) \in C^\infty. \tag{2.3}$$

With this in mind, we require that the individual is not so impatient so as to forego an infinite future of the best outcome c^* in favor of consuming c^* now.

Assumption 3 (Patience). $(c_*, c^*, c^*, c^*, \ldots) \succeq (c^*, c_*, c_*, c_*, \ldots)$.

In the context of the standard model in (1.1), patience holds if and only if $\beta \geq \frac{1}{2}$. We are ready to state our main result.

Lemma 2. *If a preference relation $\succeq$ on C^∞ satisfies continuity, stationarity, and patience, then every continuous utility function $U: C^\infty \to \mathbb{R}$ of $\succeq$ has a convex range.*

Proof. Fix a continuous utility function $U: C^\infty \to \mathbb{R}$, which, as noted previously, exists by a result of Debreu [6]. Given $c \in C$, write c^∞, as well as $(c)^\infty$, for the constant sequence $(c, c, \ldots) \in C^\infty$, and let $U^* := U((c^*)^\infty), U_* := U(c_*^\infty)$. From (2.3), we know that $U(C^\infty) \subset [U_*, U^*]$. Fix $k \in (U_*, U^*)$, and define the sequence $(c_0, c_1, \ldots) \in C^\infty$ by setting

$$c_0 = \begin{cases} c^* \text{ if } \ U(c^*, c_*^\infty) < k, \\ c_* \text{ else} \end{cases}$$

and then, inductively, for all $t > 0$,

$$c_t = \begin{cases} c^* \text{ if } U(c_0, \ldots, c_{t-1}, c^*, c_*^\infty) < k, \\ c_* \text{ else} \end{cases}$$

Because $k > U_*$ and utility is continuous, there is some t such that $c_t = c^*$. If there is a largest such t, then

$$U(c_0, c_1, \ldots) = U(c_0, \ldots, c_{t-1}, c_t, c_*^\infty) < k,$$

where the inequality follows by construction, because $c_t = c^*$. If there is no largest t such that $c_t = c^*$, then $(c_0, c_1, \ldots)$ has a subsequence $(c_{t_0}, c_{t_1}, \ldots,)$ such that $c_{t_n} = c^*$ for each n and, hence,

$$U(c_0, c_1, \ldots, c_{t_n}, c_*^\infty) < k \quad \forall n.$$

Letting $n \to \infty$, we obtain $U(c_0, c_1, \ldots) \leq k$. Next, because $U(c_0, c_1, \ldots) \leq k < U^*$, there is t such that $c_t = c_*$. If there is a largest such t, then $(c_t, c_{t+1}, c_{t+2}, \ldots) = (c_*, (c^*)^\infty)$ for some t and, hence,

$$U(c_0, c_1, \ldots) = U(c_0, \ldots, c_{t-1}, c_*, (c^*)^\infty) \leq k \leq U(c_0, \ldots, c_{t-1}, c^*, c_*^\infty), \tag{2.4}$$

where the first inequality was already established and the second follows because $c_t = c_*$. From patience, we know that $U(c_*, (c^*)^\infty) \geq U(c^*, c_*^\infty)$. Given this, a repeated application of stationarity implies that

$$U(c_0, \ldots, c_{t-1}, c_*, (c^*)^\infty) \geq U(c_0, \ldots, c_{t-1}, c^*, c_*^\infty). \tag{2.5}$$

Together, (2.4) and (2.5) imply that

$$U(c_0, c_1, \ldots) = U(c_0, \ldots, c_{t-1}, c_*, (c^*)^\infty) = U(c_0, \ldots, c_{t-1}, c^*, c_*^\infty) = k.$$

Alternatively, suppose $(c_0, c_1, \ldots)$ has a subsequence $(c_{t_0}, c_{t_1}, \ldots,)$ such that $c_{t_n} = c_*$ for all n. By construction, $U(c_0, c_1, \ldots, c_{t_n - 1}, c^*, c_*^\infty) \geq k$ for all n, and, hence,

$$U(c_0, c_1, \ldots) = \lim_n U(c_0, c_1, \ldots, c_{t_n - 1}, c^*, c_*^\infty) \geq k.$$

Because $U(c_0, c_1, \ldots) \leq k$ was already established, we are done. □

Remark 1. The proof of Lemma 2 shows that for every sequence $(c_0, c_1, \dots) \in C^\infty$, there is a sequence $(c'_0, c'_1, \dots) \in \{c_*, c^*\}^\infty$ such that $(c_0, c_1, \dots) \sim (c'_0, c'_1, \dots)$. This observation will be useful later on.

A natural weakening of patience is to require that $(c, c', c', \dots) \succ (c', c, c, \dots)$ for some $c, c' \in C$ such that $(c', c', \dots) \succ (c, c, \dots)$. As we discuss in Section 3.2, this assumption was used in Montiel Olea and Strzalecki [19] to a related but different effect. It is clear that if we invoke the assumption presently, we can still use our lemma to show that the range of U has a connected component. Some of our subsequent results, such as the uniqueness of the discount factor in an Uzawa [24] model, can be similarly "localized." To keep this paper crisp, we have not pursued this extension in detail.

As remarked in the introduction, Lemma 2 generalizes a result by Rényi [21], which we state as Corollary 1.

Corollary 1. *For every $\beta \in [\frac{1}{2}, 1)$ and every $k \in [0,1]$, there is a sequence $(u_0, u_1, \dots) \in \{0,1\}^\infty$ such that $(1-\beta)\sum_t \beta^t u_t = k$.*

In the number-theoretic literature, the sequence $(u_0, u_1, \dots)$ in Corollary 1 is known as a β-expansion of the number k. Such expansions, which need not be unique, can be computed by a number of algorithms. The proof of Lemma 2 adapts one such algorithm, due to De Vries and Komornik [5], to the more general setting of our paper. Interestingly, in Section 3.2, we use the same algorithm to develop a comparative notion of patience appropriate for the discrete choice setting under consideration.

3. The Uzawa Model

To illustrate the usefulness of Lemma 2, we now consider the Uzawa [24] model of endogenous discounting, which is one of the most commonly used classes of stationary preferences. It arises when the function ϕ in the recursive representation in (2.2) is linear in its second argument. Specifically,

$$U(c_0, c_1, c_2, \dots) = u(c_0) + b(c_0)u(c_1) + b(c_0)b(c_1)u(c_2) + \dots \tag{3.1}$$
$$= u(c_0) + b(c_0)U(c_1, c_2, \dots). \tag{3.2}$$

Above, $u : C \to \mathbb{R}$ is the instantaneous utility function, whereas the function $b : C \to (0,1)$ can be interpreted as an endogenous discount factor. We use (u, b) to denote (i) a pair of such functions, (ii) the utility function $U : C^\infty \to \mathbb{R}$ defined by them via (3.1), and (iii) the preference relation $\succeq$ on C^∞ defined by U via (2.1). When we want to make the function $U : C^\infty \to \mathbb{R}$ explicit, we may also write (u, b, U) instead of (u, b).

We note that the standard model in (1.1) is a special case and arises when the function $b : C \to (0,1)$ is constant, in which case we say that discounting is exogenous. The added flexibility of the Uzawa model is that it allows the rate at which an agent discounts the future to depend on the consumption path. Epstein and Hynes [9] discuss the value added of that in the context of a standard growth model (with divisible consumption).

Assuming patience, the next result considers the uniqueness of an Uzawa representation.

Theorem 1. *Suppose a preference relation $\succeq$ on C^∞ has two Uzawa representations, (u, b, U) and $(\hat{u}, \hat{b}, \hat{U})$. If $\succeq$ satisfies patience, then $b = \hat{b}$ and there are constants $\alpha \in \mathbb{R}_{++}, \gamma \in \mathbb{R}$ such that $U(c, c, \dots) = \alpha \hat{U}(c, c, \dots) + \gamma$ for all $c \in C$.*

Before we prove the theorem, we note that there are several equivalent ways to state the obtained uniqueness results. One is that $b = \hat{b}$ and $U(c_0, c_1, \dots) = \alpha \hat{U}(c_0, c_1, \dots) + \gamma$ for *all* sequences $(c_0, c_1, \dots) \in C^\infty$, not just constant ones. In fact, our proof is of this stronger statement. Using the fact that $U(c, c, \dots) = u(c)(1 - b(c))^{-1}$, a third way is that $b = \hat{b}$ and $u(c) = \alpha \hat{u}(c) + \gamma(1 - b(c))$ for all $c \in C$. We chose the statement we did because, as will become clear from the analysis in Section 3.1, the function $c \longmapsto U(c, c, \dots)$ and its cardinal uniqueness (i.e., its uniqueness up to positive affine transformations) is what captures the agent's desire to smooth consumption across time.

Readers familiar with properties of the standard model in (1.1) may also be puzzled by the fact that, when discounting is endogenous, the instantaneous utility functions u and $\hat{u}$ are not cardinally unique. In fact, they need not be increasing transformations of one another. This issue is not specific to the discrete choice setting; our uniqueness results are equivalent to those obtainable with a connected outcome space C. See theorem 2 in Bommier et al. [4]. The reason lies in the fact that the Uzawa model is not additively separable across time. The ensuing complementarities across time mean that one cannot perfectly dissociate the utility of a single outcome c_t from the rest of the sequence.[5] On the other hand, because all complementarities in the Uzawa model arise via the discount factor $b : C \to (0,1)$, the function $c \longmapsto U(c, c, \dots)$, which incorporates that discount factor, *is* cardinally unique.

Proof. First, note that if (u, b) represents $\succeq$, then for any $\alpha \in \mathbb{R}_{++}$ and $\gamma \in \mathbb{R}$, so does the pair $(\alpha u + \gamma(1 - b), b)$. Moreover, $(\alpha u + \gamma(1 - b), b)$ gives rise to the utility function $\alpha U + \gamma$. By choosing the parameters α and γ judiciously, we can thus assume that (u, b) is such that

$$U(c_*, c_*, \dots) = 0 \quad \textit{and} \quad U(c^*, c^*, \dots) = 1.$$

Assume that the pair $(\hat{u}, \hat{b})$ is similarly chosen, and let f be the strictly increasing function such that $f \circ U = \hat{U}$. By Lemma 2 and the normalizations we imposed, f maps $[0,1]$ onto $[0,1]$. Because the normalizations were attained using affine transformations, it is enough to show that $b = \hat{b}$ and that f is the identity function on $[0,1]$. Let $\beta := b(c_*), \delta := b(c^*), \hat{\beta} := \hat{b}(c_*), \hat{\delta} := \hat{b}(c^*)$. For every $c \in C$ and $k \in [0,1]$, we have

$$f((1 - b(c))U(c, c, \dots) + b(c)k) = (1 - \hat{b}(c))f(U(c, c, \dots)) + \hat{b}(c)f(k). \tag{3.3}$$

Taking $c = c_*$ and $c = c^*$, we deduce that for every $k \in [0,1]$,

$$f(\beta k) = \hat{\beta} k \quad \text{and} \quad f(\delta k + (1 - \delta)) = \hat{\delta} k + (1 - \hat{\delta}). \tag{3.4}$$

The first equation implies that

$$f(l) = \frac{\hat{\beta}}{\beta} l \qquad \forall l \in (0, \beta). \tag{3.5}$$

The second equation in (3.4) implies that

$$f(l) = (1 - \hat{\delta}) - \frac{\hat{\delta}}{\delta}(1 - \delta) + \frac{\hat{\delta}}{\delta} l \qquad \forall l \in [1 - \delta, 1]. \tag{3.6}$$

Given our normalizations, patience is equivalent to $\beta \geq 1 - \delta$ (and $\hat{\beta} \geq 1 - \hat{\delta}$). Hence,

$$[0, \beta] \cup [1 - \delta, 1] = [0, 1]. \tag{3.7}$$

Next, we claim that $\beta = \hat{\beta}$ and $\delta = \hat{\delta}$. Given (3.5)–(3.7), this will also show that f is the identity function on $[0,1]$. First, suppose $\beta > 1 - \delta$. For every $k \in (0,1)$ such that $1 - \delta < (1 - \delta) + \delta k < \beta$, deduce from (3.4) that

$$k(\delta\hat{\beta} - \hat{\delta}\beta) = \beta(1 - \hat{\delta}) - \hat{\beta}(1 - \delta). \tag{3.8}$$

This equality can hold for a range of k if and only if both $\delta\hat{\beta} - \hat{\delta}\beta$ and $\beta(1 - \hat{\delta}) - \hat{\beta}(1 - \delta)$ are equal to zero. But this is possible if and only if $\beta = \hat{\beta}$ and $\delta = \hat{\delta}$. Alternatively, suppose $\beta = 1 - \delta$. This is equivalent to $(c^*, c_*, c_*, \dots) \sim (c_*, c^*, c^*, \dots)$ and, hence, to $\hat{\beta} = 1 - \hat{\delta}$. Plugging $k = \beta = 1 - \delta$ into (3.8) and simplifying, deduce that $\beta = \hat{\beta}$. Under the case we are considering, this implies that $\delta = \hat{\delta}$.

It remains to show that $b(c) = \hat{b}(c)$ for any $c \in C$, not just $c \in \{c_*, c^*\}$. Because f is the identity function, we deduce from (3.3) that for every $c \in C$,

$$U(c, c, \dots)(\hat{b}(c) - b(c)) = (b(c) - \hat{b}(c))k \quad \forall k \in [0,1].$$

But this can hold if and only if $b(c) = \hat{b}(c)$. □

If discounting is exogenous, that is, $b : C \to (0,1)$ is constant, the cardinal uniqueness of $c \mapsto U(c, c, \dots)$ reduces to the cardinal uniqueness of $u : C \to \mathbb{R}$, with Theorem 1 delivering the more familiar uniqueness properties of the standard additively separable model in (1.1).[6] We summarize these properties in the next corollary. There, and subsequently, we signify that discounting is exogenous by writing β instead of b.

Corollary 2. *Suppose a preference relation $\succeq$ on C^∞ has two representations (u, β) and $(\hat{u}, \hat{\beta})$. If $\succeq$ satisfies patience, then $\beta = \hat{\beta}$ and there are constants $\alpha \in \mathbb{R}_{++}, \gamma \in \mathbb{R}$ such that $u = \alpha\hat{u} + \gamma$.*

Our next step is to flesh out our uniqueness results with some concrete behavioral implications.

3.1. Interpreting the Cardinal Uniqueness of *U*

When preferences are standard and consumption an infinitely divisible good, the curvature of $u : C \to \mathbb{R}$ captures the agent's desire to smooth consumption across time. This explains why $u : C \to \mathbb{R}$ is unique only up to affine transformations. The concept of curvature is of course ill defined when C is discrete. There is, however, a comparative static one can hope to replicate. In the standard setting, taking a concave transformation of u increases the desire to smooth consumption. This section derives an analogous result for the more general Uzawa model and the case of discrete outcomes.[7] Consistent with the discussion following Theorem 1, note that once discounting is endogenous, the object of analysis shifts from $u : C \to \mathbb{R}$ to $c \mapsto U(c, c, \dots)$.

To isolate the desire to smooth consumption, we need to assume that all other components of preference are the same. Such an assumption is standard in the type of comparative static we are after. Presently, it means that we focus on Uzawa preferences that share the same discount factor $b: C \longrightarrow (0,1)$ and the same ranking of constant sequences.[8]

Finally, we have to assume that $|C| > 2$. To see why this is necessary, note that with only two outcomes, c_* and c^*, the cardinal uniqueness of the function $c \longmapsto U(c,c,\dots)$ becomes vacuous; namely, the function $c \longmapsto U(c,c,\dots)$ becomes just a pair of numbers, $U(c_*,c_*,\dots)$ and $U(c^*,c^*,\dots)$, and because affine transformations have two degrees of freedom, we can transform that pair into any other pair $\hat{U}(c_*,c_*,\dots)$ and $\hat{U}(c^*,c^*,\dots)$. The next lemma is one way to summarize this discussion.[9]

Lemma 3. *Suppose* $|C| = 2$ *and* (u, b) *and* $(\hat{u}, b)$ *induce the same ranking on the space of constant sequences* $(c,c,\dots)$. *Then* (u, b) *and* $(\hat{u}, b)$ *induce the same ranking on the space* C^∞ *of all sequences.*

Next, we formalize what it means for one preference on C^∞ to exhibit a greater desire to smooth consumption.

Definition 1. Given two preference relations $\succeq$ and $\hat{\succeq}$ on C^∞, say that $\succeq$ exhibits a greater desire to smooth consumption if for every outcome $c \in C$ and every sequence $(c_0, c_1, \dots) \in C^\infty$,

$$(c,c,\dots) \,\hat{\succeq}\, (c_0,c_1,\dots) \quad \text{implies} \quad (c,c,\dots) \succeq (c_0,c_1,\dots). \tag{3.9}$$

The desire is *strict* if for some $c \in C$ and some sequence $(c_0,c_1,\dots) \in C^\infty$, we have

$$(c,c,\dots) \,\hat{\sim}\, (c_0,c_1,\dots) \quad \text{and} \quad (c,c,\dots) \succ (c_0,c_1,\dots). \tag{3.10}$$

In words, Definition 1 posits that whenever $\hat{\succeq}$ prefers a constant sequence, then so does the preference relation $\succeq$ that values smoothness at least as much. We should make a key point regarding the "strict" part of the definition. In principle, when outcomes are discrete, a preference relation $\hat{\succeq}$ on C^∞ may be continuous, yet fail to exhibit an indifference such as the one posited by (3.10).[10] In such a case, our test for detecting a strictly greater desire to smooth consumption would not work. Luckily, as Remark 1 shows, this case does not arise under stationarity and patience, the working assumptions of this paper. With this in mind, we can state our next theorem.

Theorem 2. *Suppose* $|C| > 2$ *and let* (u, b, U) *and* $(\hat{u}, b, \hat{U})$ *be two Uzawa preferences that satisfy patience. If there is a (strictly) concave and strictly increasing function* $f: \mathbb{R} \to \mathbb{R}$ *such that* $U(c,c,\dots) = f(\hat{U}(c,c,\dots))$ *for every* $c \in C$, *then* (u, b, U) *exhibits a (strictly) greater desire to smooth consumption.*

Proof. Take a sequence $(c_0, c_1, \dots) \in C^\infty$, and let $S = \{c^1, c^2, \dots, c^m\} \subset C$ be the set of outcomes that appear in the sequence. We claim that there is a vector $(p^1, \dots, p^m) \in [0,1]^m$ such that $\sum_i p_i = 1$ and

$$U(c_0,c_1,\dots) = \sum_{i=1}^{m} p^i U(c^i,c^i,\dots) \quad \text{and} \quad \hat{U}(c_0,c_1,\dots) = \sum_{i=1}^{m} p^i \hat{U}(c^i,c^i,\dots). \tag{3.11}$$

The notable fact is that the weights p_i are the same for both preferences. This follows from the assumption that $b = \hat{b}$. For example,

$$\begin{aligned} U(c,c',c',\dots) &= (1-b(c))U(c,c,\dots) + b(c)U(c',c',\dots), \quad \text{and} \\ \hat{U}(c,c',c',\dots) &= (1-b(c))\hat{U}(c,c,\dots) + b(c)\hat{U}(c',c',\dots). \end{aligned}$$

By induction, the proof holds for every sequence $(c_0,c_1,\dots)$ such that for some t, $c_k = c_t$ for all $k \geq t$. Call such sequences *finite*. For an arbitrary sequence $(c_0,c_1,\dots)$, let $(t_1^i, t_2^i, \dots)$ be the sequence of time periods t such that $c_t = c^i$. In addition, let

$$p_{t_k^i} := \begin{cases} (1-b(c^i)) \text{ if } t_k^i = 0, \\ (1-b(c^i))\Pi_{\tau=0}^{t_k^i - 1} b(c_\tau) \text{ if } t_k^i > 0; \end{cases}$$

$$p^i := \sum_k p_{t_k^i}.$$

If c^i appears finitely many times in $(c_0,c_1,\dots)$, then p^i is a finite sum and hence well defined. If c^i appears infinitely many times, then the sum p^i is well defined by the ratio test, because

$$\frac{p_{t_{k+1}^i}}{p_{t_k^i}} = \prod_{\tau = t_k^i}^{t_{k+1}^i - 1} b(c_\tau) \leq \max_i b(c^i) < 1.$$

It remains to show that $\sum_i p^i = 1$. We argued that this is true when $(c_0,c_1,\dots)$ is finite. Because any sequence can be approximated by finite sequences, the same is true for any $(c_0,c_1,\dots)$.

Next, take some $c \in C$ and $(c_0, c_1, \ldots) \in C^\infty$ such that $\hat{U}(c, c, \ldots) \geq \hat{U}(c_0, c_1, \ldots)$. Then,

$$U(c_0, c_1, ..) = \sum_i p^i U(c^i, c^i, \ldots) = \sum_i p^i f(\hat{U}(c^i, c^i, \ldots)) \tag{3.12}$$

$$\leq f\left(\sum_i p^i \hat{U}(c^i)\right) = f(\hat{U}(c_0, c_1, \ldots)) \leq f(\hat{U}(c, c, \ldots)) = U(c, c, \ldots). \tag{3.13}$$

Finally, suppose f is strictly concave, and let $c \in C$ be such that $\hat{U}(c^*, c^*, \ldots) > \hat{U}(c, c, \ldots) > \hat{U}(c_*, c_*, \ldots)$. From Remark 1, we know that there is a sequence $(c_0, c_1, \ldots) \in \{c_*, c^*\}^\infty$ such that $\hat{U}(c_0, c_1, \ldots) = \hat{U}(c, c, \ldots)$. By construction, the first inequality in (3.12) must be strict. □

If we strengthen patience, we can also obtain a converse of Theorem 2, whereby a greater desire to smooth consumption implies a "more concave" utility.[11] Consider the following axiom.

Assumption 4 (Strong Patience). *If $c, c' \in C$ are such that $(c', c', \ldots) \succ (c, c, \ldots)$, then $(c, c', c', \ldots) \succeq (c', c, c, \ldots)$.*

Within the Uzawa model, strong patience holds if and only if $b(c) + b(c') \geq 1$ for all $c \neq c'$.[12] By comparison, patience holds if and only if $b(c_*) + b(c^*) \geq 1$. The axioms are equivalent when discounting is exogenous.

Theorem 3. *Suppose $|C| > 2$, and let (u, b, U) and $(\hat{u}, b, \hat{U})$ be two Uzawa preferences that satisfy strong patience. If (u, b, U) exhibits a greater desire to smooth consumption than $(\hat{u}, b, \hat{U})$, then there is a concave and strictly increasing function $f: \mathbb{R} \to \mathbb{R}$ such that $U(c, c, \ldots) = f(\hat{U}(c, c, \ldots))$ for every $c \in C$. In addition, if the desire is strict, then f is nonlinear.*[13]

We begin the proof with a lemma showing that if two stationary preference relations are comparable under Definition 1, they must share the same ranking of constant sequences.

Lemma 4. *Let $\succeq, \hat{\succeq}$ be two stationary and continuous preference relations on C^∞. If $\succeq$ exhibits a greater desire to smooth consumption than $\hat{\succeq}$, then $\succeq$ and $\hat{\succeq}$ induce the same ranking on the set of constant sequences.*

Proof. Under the simplifying assumption that no distinct constant sequences are indifferent, the proof of the lemma follows directly from (3.9). We show that the lemma holds without this assumption. From (3.9), we know that $(c, c, \ldots) \hat{\succeq} (c', c', \ldots)$ implies $(c, c, \ldots) \succeq (c', c', \ldots)$. Hence, if $(c^*, c^*, \ldots), (c_*, c_*, \ldots)$ are a best and, respectively, worst sequence for $\hat{\succeq}$, they are also a best and, respectively, worst sequence for $\succeq$. Suppose for some $c, c' \in C$, $(c, c, \ldots) \hat{\succ} (c', c', \ldots)$ and $(c, c, \ldots) \sim (c', c', \ldots)$. Because we rule out complete indifference (see Endnote 3), we must have

$$(c, c, \ldots) \sim (c', c', \ldots) \succ (c_*, c_*, \ldots) \quad \text{or} \quad (c^*, c^*, \ldots) \succ (c, c, \ldots) \sim (c', c', \ldots).$$

Assume the latter. The other case follows from similar arguments. From (3.9), it follows that $(c^*, c^*, \ldots) \hat{\succ} (c, c, \ldots)$ and, by the choice of c and c', that

$$(c^*, c^*, \ldots) \hat{\succ} (c, c, \ldots) \hat{\succ} (c', c', \ldots).$$

Let $s^t \in C^\infty$ be the sequence that gives c' during the first t periods and c^* thereafter. Because $\hat{\succeq}$ is continuous, we know that $(c, c, \ldots) \hat{\succ} s^t$ for all t large enough. At the same time, by stationarity and the case we consider, $s^t \succ (c', c', \ldots) \sim (c, c, \ldots)$ for all t. Together, the last two observations contradict (3.9). □

Lemma 4 raises the question of whether comparability under Definition 1 is strong enough to imply that two Uzawa preferences share the same discount factors $b: C \to (0, 1)$. The next example shows that this is not the case.

Example 1. Suppose $C = \{c_*, c, c^*\}$, and let $\succeq$ and $\hat{\succeq}$ have standard representations (u, β) and $(\hat{u}, \hat{\beta})$ such that $u(c_*) = \hat{u}(c_*) = 0$, $u(c) = 1 - \beta, \hat{u}(c) = 1 - \hat{\beta}$, $u(c^*) = \hat{u}(c^*) = 1$ and $\beta > \hat{\beta} = 0.5$. Then, $\succeq$ exhibits a greater desire to smooth consumption, but u is not a concave transformation of $\hat{u}$.

We are ready to complete the proof of Theorem 3.

Proof of Theorem 3. It will be convenient to write U_c for $U(c, c, \ldots)$, with $\hat{U}_c$ similarly defined. By Lemma 4, there is a strictly increasing function $f: \{\hat{U}_c : c \in C\} \to \{U_c : c \in C\}$. Though the domain of f is finite, we can identify f with a piecewise linear function $f: \hat{U}(C^\infty) \to U(C^\infty)$ in the obvious way. Let $\underline{c}, c, \bar{c} \in C$ be such that $U_{\bar{c}} > U_c > U_{\underline{c}}$. We claim that there is $\lambda \in (0, 1)$ such that $f(\lambda \hat{U}_{\bar{c}} + (1 - \lambda)\hat{U}_{\underline{c}}) \geq \lambda f(\hat{U}_{\bar{c}}) + (1 - \lambda) f(\hat{U}_{\underline{c}})$. Given the piecewise linear nature of f, this will imply that f is concave. Strong patience and Lemma 2 imply that there is a sequence $(\hat{c}_0,\hat{c}_1,\dots)\in\{\underline{c},\overline{c}\}^\infty$ such that $\hat{U}(\hat{c}_0,\hat{c}_1,\dots)=\hat{U}_c$. In addition, using (3.11), there exists $\lambda\in(0,1)$ such that

$$U(\hat{c}_0,\hat{c}_1,\dots)=\lambda U_{\overline{c}}+(1-\lambda)U_{\underline{c}} \quad\text{and}\quad \hat{U}_c=\hat{U}(\hat{c}_0,\hat{c}_1,\dots)=\lambda\hat{U}_{\overline{c}}+(1-\lambda)\hat{U}_{\underline{c}}.$$

Because $\succeq$ exhibits a greater desire to smooth consumption, $\hat{U}_c=\hat{U}(\hat{c}_0,\hat{c}_1,\dots)$ implies $U_c\ge U(\hat{c}_0,\hat{c}_1,\dots)=\lambda U_{\overline{c}}+(1-\lambda)U_{\underline{c}}$. Putting everything together gives

$$f(\lambda\hat{U}_{\overline{c}}+(1-\lambda)\hat{U}_{\underline{c}})=f(\hat{U}_c)=U_c\ge\lambda U_{\overline{c}}+(1-\lambda)U_{\underline{c}}=\lambda f(\hat{U}_{\overline{c}})+(1-\lambda)f(\hat{U}_{\underline{c}}).$$

Thus, f is concave. Finally, a linear f would imply that $\succeq\;=\;\hat{\succeq}$, which would make (3.10) impossible. □

3.2. Uniqueness of the Discount Factor: Some Implications

Discount factors are typically defined as the inverse of the marginal rate of substitution between two successive outcomes, c_t and c_{t+1}, along a constant sequence $(c,c,\dots)$. In the context of the Uzawa model, this marginal rate of substitution depends on c, but not on t, and is equal to $b(c)^{-1}$ (see Epstein [8, p. 137]). In the context of the standard model, the marginal rate of substitution is independent of c, as well as t, and is equal to β^{-1}.

Unfortunately, this ordinal definition of discount factors is not available when the outcome space C is discrete.[14] At the same time, the uniqueness result in Theorem 1 strongly suggests that $b:C\to(0,1)$ continues to have a well-defined meaning in terms of behavior. This section presents two results that help clarify this meaning: a comparative static and, if discounting is exogenous, a way to measure β up to any level of precision.

Let $\succeq$ and $\hat{\succeq}$ be two Uzawa preferences on C^∞ with respective representations (u,b) and $(\hat{u},\hat{b})$. Assume that $\succeq$ and $\hat{\succeq}$ share the same best and worst sequences, $(c^*,c^*,\dots)$ and $(c_*,c_*,\dots)$, and both satisfy patience. Assume further that $b(c_*)\ge\hat{b}(c_*)$ and $b(c^*)\ge\hat{b}(c^*)$, with at least one inequality strict. When attention is restricted to sequences in $\{c_*,c^*\}^\infty$, it is natural to interpret these inequalities as $\succeq$ being more patient than $\hat{\succeq}$. Our first result delivers a behavioral implication in support of this interpretation.

Before we jump into the details, we note that the identity of c^* and c_* as best and worst is immaterial. Everything goes through for arbitrary outcomes $c\ne c'\in C$ as long as $\succeq$ and $\hat{\succeq}$ agree on the ranking of $(c,c,\dots)$ and $(c',c',\dots)$ and an analogue of patience holds.

To proceed, focus on $\succeq$. The proof of Lemma 2 gives an algorithm for constructing a sequence $(c_0,c_1,\dots)\in\{c_*,c^*\}^\infty$ such that $c_0=c_*$ and $(c_0,c_1,\dots)\sim(c^*,c_*,c_*,\dots)$; namely, for each $t>0$,

$$c_t=\begin{cases}c^* & \text{if}\quad (c^*,c_*,c_*,\dots)\succ(c_*,c_1,c_2,\dots,c_{t-1},c^*,c_*,c_*,\dots),\\ c_* & \text{otherwise.}\end{cases}\tag{3.14}$$

In words, the algorithm constructs $(c_0,c_1,\dots)$ by trying to squeeze in as many c^* as early as possible, but not in $t=0$, without generating a preference for $(c_0,c_1,\dots)$ over $(c^*,c_*,c_*,\dots)$.[15]

Intuitively, a less patient agent will require more of c^* early on to compensate for the fact that c^* is not consumed in $t=0$. To make this precise, let $(\hat{c}_0,\hat{c}_1,\dots)\in\{c_*,c^*\}^\infty$ be the analogue of $(c_0,c_1,\dots)$ given the preference relation $\hat{\succeq}$. Say that $(\hat{c}_0,\hat{c}_1,\dots)$ *lexicographically dominates* $(c_0,c_1,\dots)$ if $(\hat{c}_0,\hat{c}_1,\dots)$ is the first sequence to deliver a c^*-outcome.

Theorem 4. *If $b(c_*)\ge\hat{b}(c_*)$ and $b(c^*)\ge\hat{b}(c^*)$, with at least one inequality strict, then $(\hat{c}_0,\hat{c}_1,\dots)$ lexicographically dominates $(c_0,c_1,\dots)$.*

Proof. It is w.l.o.g. to assume that $u(c_*)=\hat{u}(c_*)=0$ and $u(c^*)=\hat{u}(c^*)=1$. If $(\hat{c}_0,\hat{c}_1,\dots)$ does not lexicographically dominate $(c_0,c_1,\dots)$, then either $(c_0,c_1,..)$ lexicographically dominates $(\hat{c}_0,\hat{c}_1,\dots)$ or the two sequences are equal. Suppose $(c_0,c_1,..)$ dominates $(\hat{c}_0,\hat{c}_1,\dots)$, and let $N\ge1$ be the first instance of $c_N=c^*$ and $\hat{c}_N=c_*$. Hence, $\hat{c}_t=c_t$ for all $t<N$ and

$$(c_0,c_1,\dots,c_{N-1},c^*,c_*,c_*,\dots)\prec(c^*,c_*,c_*,\dots)\quad\text{and}$$

$$(c_0,c_1,\dots,c_{N-1},c^*,c_*,c_*,\dots)\;\hat{\succeq}\;(c^*,c_*,c_*,\dots).$$

This is equivalent to

$$\sum_{t=0}^{N-1}\prod_{j=0}^{t-1}b(c_j)u(c_j)+\prod_{j=0}^{N-1}b(c_j)<1\quad\text{and}\quad\sum_{t=0}^{N-1}\prod_{j=0}^{t-1}\hat{b}(c_j)\hat{u}(c_j)+\prod_{j=0}^{N-1}\hat{b}(c_j)\ge1.$$

Combining these inequalities with $u = \hat{u} \geq 0$ on $\{c_*, c^*\}$ and $b \geq \hat{b}$ gives a contradiction. Alternatively, suppose $(\hat{c}_0, \hat{c}_1, \dots) = (c_0, c_1, \dots)$. Then,

$$(c_0, c_1, \dots) \sim (c^*, c_*, c_*, ..) \quad \text{and} \quad (c_0, c_1, \dots)\hat{\sim}(c^*, c_*, c_*, ..).$$

Equivalently,

$$\sum_{\{t: c_t = c^*\}} \prod_{j=0}^{t-1} b(c_j) = 1 = \sum_{\{t: c_t = c^*\}} \prod_{j=0}^{t-1} \hat{b}(c_j). \tag{3.15}$$

Because $b \geq \hat{b}$ on $\{c_*, c^*\}$, this is possible only if $\prod_{j=0}^{t-1} b(c_j) = \prod_{j=0}^{t-1} \hat{b}(c_j)$ for each t such that $c_t = c^*$, which we claim cannot be true. Indeed, recall that $c_0 = c_*$. Using the Uzawa representation, deduce that $(c^*, c_*, c_*, \dots) \succ (c_*, c^*, c_*, c_*, \dots)$ and hence that $c_1 = c^*$. By continuity, the ranking $(c^*, c_*, c_*, \dots) \succ (c_*, c^*, c_*, c_*, \dots)$ will not change if one of the c_* in the sequence $(c_*, c^*, c_*, c_*, \dots)$, sufficiently far in time, is changed to c^*. It follows that $c_t = c^*$ for some $t > 1$. But then the product $\prod_{j=0}^{t-1} b(c_j)$ contains both $b(c_*)$ and $b(c^*)$, implying that $\prod_{j=0}^{t-1} b(c_j) > \prod_{j=0}^{t-1} \hat{b}(c_j)$. □

Unfortunately, Theorem 4 does not tell us how to pin down the specific values taken by b and $\hat{b}$ from behavior. This changes if we assume that discounting is exogenous. Then, the indifference $(c_0, c_1, \dots) \sim (c^*, c_*, c_*, \dots)$ used in the proof of Theorem 4 identifies $\beta \in (0, 1)$ uniquely, and the algorithm used to construct $(c_0, c_1, \dots)$ can be used to compute β up to any level of precision. To confirm this, normalize utility so that $u(c^*) = 1$ and $u(c_*) = 0$. Expressing the indifference $(c_0, c_1, \dots) \sim (c^*, c_*, c_*, \dots)$ in utility terms gives $\sum_t \beta^t u(c_t) = 1$. Because $(c_0, c_1, \dots) \in \{c_*, c^*\}^\infty$, each $u(c_t)$ is either zero or one. Thus, one can view $\sum_t \beta^t u(c_t) = 1$ as an equation with a single unknown: β. Because the sum $\sum_t \beta^t u(c_t)$ is strictly increasing in β, the equation has a unique solution.[16]

In practice, one can only compute a finite truncation $(c_0, c_1, \dots, c_t)$ of the infinite sequence $(c_0, c_1, \dots)$. This, however, is enough to get an "estimate" β_t of β which converges to β as the length t of the truncation goes to infinity. To see this, take $(c_0, c_1, \dots, c_t)$ and define the function

$$f_t(\lambda) = \sum_{k=0}^{t} \lambda^k u(c_k).$$

We claim that there is a unique $\lambda \in \left[\frac{1}{2}, 1\right]$ such that $f_t(\lambda) = 1$. By construction, $c_0 = c_*$ and $c_1 = c^*$. Because $u(c_*) = 0$, we see that $f_t\left(\frac{1}{2}\right) < 1$. Because $u(c^*) = 1$, we see that $f_t(1) \geq 1$. Because f_t is strictly increasing in λ, there is a unique λ such that $f_t(\lambda) = 1$. Let β_t be that λ. Repeating the construction for each t gives us a decreasing sequence $(\beta_t)_t$ such that $\beta_t \geq \beta$ for every t. To show that $\beta = \lim_t \beta_t$, let $\delta := \lim_t \beta_t$ and note that

$$1 \geq \sum_{k=0}^{t} \beta_t^k u(c_k) \geq \sum_{k=0}^{t} \delta^k u(c_k) \geq \sum_{k=0}^{t} \beta^k u(c_k) \quad \forall t.$$

Taking limits as $t \to \infty$, we see that $\sum_{k=0}^{\infty} \delta^k u(c_k) = 1$. Thus, $\delta = \beta$, as desired.

We should note that the construction of the upper bounds β_t is not new and parallels the analysis in Montiel Olea and Strzalecki [19]. We have included it here because Montiel Olea and Strzalecki [19] have a different objective than ours and consequently highlight a different set of lessons. In particular, their setting is one in which consumption is divisible and the uniqueness of the discount factor can be deduced by more standard methods à la Koopmans [14]. Perhaps for this reason, Montiel Olea and Strzalecki [19] do not explicitly discuss the uniqueness of β or the fact that it can be obtained as the limit of the upper bounds β_t. Instead, their main concern is showing that one can measure β without having to first measure the curvature of the instantaneous utility function $u : C \to \mathbb{R}$. This conceptually distinct goal overlaps with ours because of an observation made in Section 3.1; namely, when dealing with only two outcomes, the curvature of utility becomes vacuous and the discount factor is all that matters. Moreover, because discounting is exogenous, it does not matter which two outcomes $c, c' \in C$ are chosen (as long as $\beta \geq \frac{1}{2}$ and $u(c) \neq u(c')$). This leads Montiel Olea and Strzalecki [19] to assume that $(c, c', c', \dots) \succ (c', c, c, \dots)$ for some $c, c' \in C$ such that $(c', c', \dots) \succ (c, c, \dots)$ (a weakening of patience) and ask whether the restriction of $\succeq$ to $\{c, c'\}^\infty$ can be used to measure β, leading them to the bounds β_t.[17]

Another reason we included the construction of the upper bounds β_t is to highlight the many uses of the algorithm behind Lemma 2. As we observed in Section 2, the algorithm originated in the number-theoretic literature where it can be used to prove Rényi's [21] result. We used the algorithm to prove Lemma 2 and, by extension, our corollaries concerning Uzawa preferences. But the present discussion shows that the algorithm is not just a technical tool to prove theorems. Applied appropriately, it delivers sequences that are intimately related to discounting. These sequences can be used to define and measure discount factors and carry out comparative statics.

4. The Necessity of Patience

Our results so far indicate that if $|C| = 2$, then the cardinal uniqueness of utility becomes vacuous, *but not* the uniqueness of the discount factor. Continuing to assume that $|C| = 2$, we now argue that patience is necessary for the uniqueness of the discount factor. In fact, without patience, there are no intertemporal trade-offs whatsoever. Specifically, we show that if $|C| = 2$ and $\succeq$ is continuous and stationary, but fails patience, then $\succeq$ is the lexicographic order according to which a sequence is better if it is the first to deliver a better outcome, whatever the future may be; that is, the future can never compensate for an inferior outcome in the present. We also show that given a lexicographic order $\succeq$ on C^∞ and *any* $\beta < \frac{1}{2}$, there is a utility function $U(c_0, c_1, \dots) = \sum_t \beta^t u(c_t)$ representing $\succeq$. In other words, when there are only two possible outcomes, low levels of patience ($\beta < \frac{1}{2}$) cannot be distinguished from one another.

To formalize these ideas, let $\succeq$ be a preference relation on C^∞, and write the set C of outcomes as $\{c_*, c^*\}$, where, as before, we assume that $(c^*, c^*, \dots) \succ (c_*, c_*, \dots)$. Say that $\succeq$ is *lexicographic* if for any two sequences $(c_0, c_1, \dots), (c'_0, c'_0, \dots) \in C^\infty$, we have $(c_0, c_1, \dots) \succ (c'_0, c'_0, \dots)$ if either $c_0 = c^*$ and $c'_0 = c_*$ or for some $t > 0$, $c_k = c'_k$ for all $k < t$ and $c_t = c^*$ and $c'_t = c_*$.

Theorem 5. *If $C = \{c_*, c^*\}$ and $\succeq$ is continuous and stationary, but fails patience, then $\succeq$ is lexicographic. In addition, for every $\beta \in \left(0, \frac{1}{2}\right)$ and every $u : \{c_*, c^*\} \to \mathbb{R}$ such that $u(c^*) > u(c_*)$, the function $U(c_0, c_1, \dots) = \sum_t \beta^t u(c_t)$ represents $\succeq$.*

Proof. To show that $\succeq$ is lexicographic, it is enough, by stationarity, to show that for any two sequences $(c^*, c_1, c_2, \dots)$ and $(c_*, c'_1, c'_2, \dots)$, we have $(c^*, c_1, c_2, \dots) \succ (c_*, c'_1, c'_2, \dots)$. Recall from Section 2 that $(c^*, c^*, \dots)$ is the best and $(c_*, c_*, \dots)$ the worst among all sequences in C^∞.[18] Together with stationarity and the failure of patience, we see that

$$(c^*, c_1, c_2, \dots) \succeq (c^*, c_*, c_*, \dots) \succ (c_*, c^*, c^*, \dots) \succeq (c_*, c'_1, c'_2, \dots).$$

Finally, it is immediate that any function $U(c_0, c_1, \dots) = \sum_t \beta^t u(c_t)$ such that $u(c^*) > u(c_*)$ and $\beta \in \left(0, \frac{1}{2}\right)$ represents $\succeq$. □

Theorem 5 shows that patience is necessary for the uniqueness result established in Theorem 1. As we now show, patience is also necessary for the conclusion of Lemma 2.

Theorem 6. *If $|C| = 2$ and $\succeq$ is a lexicographic preference relation on C^∞, then the range of any utility function $U : C^\infty \to \mathbb{R}$ of $\succeq$ is totally disconnected.*

Proof. We have to show that the range of U contains no proper interval. The key observation is that because $\succeq$ is lexicographic, there is no sequence $(c_0, c_1 \dots) \in C^\infty$ such that

$$(c^*, c_*, c_*, \dots) \succ (c_0, c_1 \dots) \succ (c_*, c^*, c^*, \dots). \tag{4.1}$$

Let $U' > U$ be two values in the range of $U : C^\infty \to \mathbb{R}$, and let $(c'_0, c'_1, \dots), (c_0, c_1, \dots) \in C^\infty$ be the sequences that generate these utility values. Because $\succeq$ is lexicographic, there is t such that $c'_k = c_k$ for all $k < t$, $c'_t = c^*$, and $c_t = c_*$. But then,

$$U' \geq U(c_0, c_1, \dots, c_{t-1}, c^*, c_*, c_*, \dots) > U(c_0, c_1, \dots, c_{t-1}, c_*, c^*, c^*, \dots) \geq U.$$

We claim that there is no $(c''_0, c''_1, \dots) \in C^\infty$ such that

$$(c_0, c_1, \dots, c_{t-1}, c^*, c_*, c_*, \dots) \succ (c''_0, c''_1, \dots) \succ (c_0, c_1, \dots, c_{t-1}, c_*, c^*, c^*, \dots).$$

If $c''_k = c_k$ for every $k < t$, the claim follows from (4.1) and the stationarity of $\succeq$. Otherwise, let k be the smallest $\tau < t$ such that $c''_\tau \neq c_\tau$. If $c''_k = c^*$ and $c_k = c_*$, then $(c''_0, c''_1, \dots) \succ (c_0, c_1, \dots, c_{t-1}, c^*, c_*, c_*, \dots)$. Similarly, if $c''_k = c_*$ and $c_k = c^*$, then $(c_0, c_1, \dots, c_{t-1}, c_*, c^*, c^*, \dots) \succ (c''_0, c''_1, \dots)$.[19] □

One can ask whether patience can be weakened if $|C| > 2$. We have not pursued this question because the analysis is quite involved and because we consider patience to be a rather weak requirement. For the same reason, we feel that the next issue is largely technical. We expound on it because it presents one case in which the simplifying assumption that all constant sequences be strictly ranked comes with some loss of generality.

To set the stage, say that a preference relation $\succeq$ on C^∞ is *binary* if there is no $c \in C$ s.t. $(c^*, c^*, \dots) \succ (c, c, \dots) \succ (c_*, c_*, \dots)$. If we allow distinct constant sequences to be indifferent, a preference $\succeq$ could be binary even when $|C| > 2$. One may then ask whether Theorems 5 and 6 carry over to the case of binary preferences. In the context of Theorem 6, we can show that a utility function of any stationary preference that is binary and fails patience has a disconnected range. On the other hand, we do not know if the range is totally disconnected. With regard to Theorem 5, the next example shows that an Uzawa preference that is binary and fails patience need not be lexicographic. In such cases, we do not know if the discount factor $b : C \to (0,1)$ is unique or not.

Example 2. Suppose $C = \{c_*, c, c^*\}$ and $\succeq$ has an Uzawa representation $(u,\ b)$ with $b(c_*) = 0.05, b(c) = 0.4, b(c^*) = 0.1$, $u(c_*) = 0, u(c) = 1 - b(c)$, and $u(c^*) = 1 - b(c^*)$. With these specifications, $(c^*, c_*, c_*, \dots)$ is strictly preferred to $(c, c, c_*, c_*, \dots)$.

Finally, we note that Theorems 5 and 6 both extend to the case of binary preferences if we make the assumption that preferences are *monotonic*, that is, that for every $c', c'' \in C$ and every $(c_0, c_1, \dots) \in C^\infty$, we have

$$(c'', c'', \dots) \succeq (c', c', \dots) \Leftrightarrow (c'', c_0, c_1, \dots) \succeq (c', c_0, c_1, \dots).$$

Preferences that admit a standard representation (u, β) are automatically monotonic but, as Example 2 confirms, Uzawa preferences need not be.

5. Conclusions and Related Literature

This paper presented a generalization of a well-known number-theoretic result by Rényi [21] and used it to investigate a standard question in decision theory, namely, the uniqueness and behavioral meaning of preference parameters. Our hope is that this generalization and the techniques behind it will set the stage for other applications. Discrete outcomes arise naturally in several settings: repeated games with discrete action spaces and no public (or private) randomization, experimental work with nonmonetary prizes, and, of course, the large empirical literature on discrete dynamic choice.

In the context of repeated games, Fudenberg and Maskin [10] and Sorin [23] use a multiagent generalization of Rényi [21] to show that the space of payoffs attainable by pure strategies is convex. Based on this, Fudenberg and Maskin [10] obtain a folk theorem without the use of public randomization. A multiagent generalization of Lemma 2 may be similarly useful in games in which preferences are recursive but not time separable. In this context, we note a small but recent literature showing that time separability, though standard in the study on repeated games, is not an innocuous assumption. See Kochov and Song [12], Sekiguchi and Wakai [22], and Obara and Park [20].

As we observed at the end of Section 3.2, a major question in experimental work has been how to measure discount factors without placing assumptions on the instantaneous utility function. One approach, proposed by Attema et al. [2], assumes a divisible consumption good. An alternative approach, proposed by Montiel Olea and Strzalecki [19], uses the richness of the time dimension, while requiring only two distinct outcomes. Both papers assume time separability. By comparison, our results regarding Uzawa preferences suggest that progress can be made even when preferences are not separable (and the consumption good is indivisible).

Building on an earlier version of this paper, Mackenzie [17] considers the uniqueness of subjective beliefs in a setting with a discrete, but infinite, state space and a discrete outcome space. Reinterpreting states as time periods, one can recast Mackenzie's [17] problem as one concerning preferences that are time separable but not stationary. By comparison, our main results concern preferences that are stationary but not time separable. The different focus of Mackenzie's [17] paper leads to different challenges, both formal and conceptual. One conceptual challenge concerns the formulation of patience. Presently, the axiom concerns only a single ranking (the one between $(c_*, c^*, c^*, \dots)$ and $(c^*, c_*, c_*, \dots)$). By invoking stationarity, however, we can ensure that patience applies in many more choice situations.[20] Obviously, the same argument does not work in Mackenzie [17]. One of his main contributions, therefore, is to propose a stronger condition that applies to a variety of trade-offs and choice situations.

Finally, consider the large empirical literature on discrete dynamic choice.[21] In that context, the presence of uncertainty and the concomitant assumption of expected utility can be used to "convexify" the space of utility outcomes. The uniqueness of preference parameters may then be deduced by standard arguments. It is well known, however, that the standard expected utility model confounds risk attitudes with the desire to smooth consumption over time.[22] If there is a reason to believe that these components of preference are distinct, it becomes beneficial to know whether the intertemporal components of a representation can be substantiated using only intertemporal behavior, without appealing to the presence of uncertainty. Our work and the techniques behind it may be helpful in such pursuits.

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Endnotes

[1] The last observation is closely related to results in Montiel Olea and Strzalecki [19]. We discuss the connection in Section 3.2.

[2] See Mas-Colell et al. [18, chapter 1B] for definitions. As is usual, we use $\sim$ and $\succ$ to denote respectively the indifference and the strict preference relation associated with $\succeq$.

[3] When the simplifying assumption that all constant sequences be strictly ranked is relaxed, we still assume that *some* sequences are strictly ranked, that is, that the preference $\succeq$ is not one of complete indifference.

[4] See, for instance, Kochov [11, lemma 3.4].

[5] An analogous problem arises in the classical theory of consumer choice (Mas-Colell et al. [18, chapter 3]), where unless utility is additively separable across goods, one cannot speak of the utility of a single good, but must instead speak of the utility of the entire bundle of goods.

[6] In the case of a connected outcome space, these properties were first obtained by Koopmans [14].

[7] We are unaware if such a comparative static for the Uzawa model has been previously developed, whether C is assumed connected or not.

[8] The assumption that preferences share the same ranking of constant sequences is not invoked directly (except in Lemma 3). Rather, it is implied (and hence implicitly assumed) whenever two preference relation are comparable under our definition of "a greater desire to smooth consumption." See Lemma 4.

[9] If we allow for indifferences among the set of constant sequences, the assumption that $|C| > 2$ must be restated as saying that there are at least three strictly ranked constant sequences, that is, that there is $c \in C$ s.t. $(c^*, c^*, \dots) \succ (c, c, \dots) \succ (c_*, c_*, \dots)$.

[10] An example is provided by the lexicographic orders discussed in Section 4.

[11] Though our strengthening is sufficient for this converse, we do not know if it is necessary.

[12] Write $U(c, c', c', \dots) \geq U(c', c, c, \dots)$ as $u(c) + b(c)U(c', c', \dots) \geq u(c') + b(c')U(c, c, \dots)$. Because $u(c'') = (1 - b(c''))U(c'', c'', \dots)$ for all c'', we get $(U(c', c', \dots) - U(c, c, \dots))(b(c') + b(c) - 1) \geq 0$. Because $U(c', c', \dots) > U(c, c, \dots)$, we are done.

[13] Because $\{\hat{U}(c, c, \dots) : c \in C\}$ is a finite set, we can also choose f to be strictly concave, by which we mean that $f(\alpha k + (1-\alpha)k') > \alpha f(k) + (1-\alpha)f(k')$ whenever $k \neq k'$ and $\alpha \in (0, 1)$. We omit the details.

[14] By ordinal, we mean that a parameter of a utility representation can be defined directly in terms of the underlying preference relation $\succeq$.

[15] For this reason, the algorithm is sometimes called a "greedy algorithm."

[16] By comparison, if discounting is endogenous, the indifference $(c_0, c_1, \dots) \sim (c^*, c_*, c_*, \dots)$ translates into a polynomial equation with two unknowns: $b(c^*)$ and $b(c_*)$. One must therefore supplement the indifference with other rankings. Finding a minimal, hopefully finite, set of such rankings is an important open question.

[17] We have tailored our discussion of Montiel Olea and Strzalecki [19] to the present setting and notation. These differences are not essential.

[18] This is the only place in the proof where continuity is used. In particular, Theorem 5 would remain true if one were to replace continuity with the weaker assumption that the best and worst sequences in C^∞ can be chosen to be constant.

[19] Though simple, Theorem 6 brings up another curious connection to the number-theoretic literature; namely, it is well known that if $\beta = \frac{1}{3}$, the discounted average $(1-\beta)\sum_t \beta^t u_t$, defined on the space of sequences $(u_0, u_1, \dots) \in \{0, 1\}^\infty$, has as its image the Cantor set, which is totally disconnected. Theorems 5 and 6 show that this is not particular to the discounted formula. What matters is the underlying lexicographic order on $\{0, 1\}^\infty$. Any utility function of that order has a totally disconnected range.

[20] For example, under stationarity, $(c_*, c^*, c^*, \dots) \succeq (c^*, c_*, c_*, \dots)$ implies $(c_0, \dots, c_t, c_*, c^*, c^*, \dots) \succeq (c_0, \dots, c_t, c^*, c_*, c_*, \dots)$ for all t and $c_0, \dots, c_t \in C$.

[21] See Aguirregabiria and Mira [1] for a recent survey.

[22] See Lu and Saito [15] for a related problem concerning the measurement of risk aversion when choices are stochastic (as in the discrete choice literature) and risk aversion is distinct from the desire to smooth consumption over time.

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