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A Computational Procedure to Deviation of a Random Signal from a Given Target Based of Piecewise Constant Admissible Controls

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Abstract

The problem of minimization of the mean square value of the deviation of a random signal from a given target is analyzed in the paper[1]. Here we introduce and explain a computational procedure to compute the gain matrices of the optimal piecewise constant admissible controls to solve this problem. We apply the procedure on several examples.

1 Introduction

We shall apply the class of admissible controls that consists of all piecewise constant stochastic processes of the form [1]

$$u(t) = u_k, \qquad t_k \le t < t_{k+1}, \qquad 0 \le k \le N-1\,, \tag{1}$$

where $t_0 < t_1 < ... < t_N = t_f$ is a partition of the interval $[t_0, t_f]$ and $u_k : \Omega \to \mathbb{R}^m$ are random vectors which are $\mathcal{F}_{t_k}$-measurable and $\mathbb{E}[|u_k|^2] < \infty$, $0 \le k \le N-1$.

In the sequel we shall denote $\mathcal{U}_{pc}$ the class of piecewise constant controls of type (1).

Consider the random signal $z(t)$, $t_0 \le t \le t_f$:

$$z(t) = C(t)x(t) \tag{2}$$

where $x(t)$ is the vector of the states of the controlled system:

$$\begin{aligned} dx(t) &= (A_0(t)x(t) + B_0(t)u_k)dt + (A_1(t)x(t) + B_1(t)u_k)dw(t), \\ &\quad t_k \le t < t_{k+1}, 0 \le k \le N-1 \\ x(t_0) &= x_0 \end{aligned} \tag{3}$$

where $u(t)$ is the vector of the control parameters. In (3) $\{w(t)\}_{t\ge 0}$ is a 1-dimensional standard Wiener process defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$, that is, $w(0) = 0$ and for each $t > 0$, $\mathbb{E}[w(t)] = 0$, $\mathbb{E}[(w(t) - w(s))^2] = t - s$ if $0 \le s \le t$, and

$$J(x_0,\zeta,u(\cdot)) = \mathbb{E}[|z_u(t_f)-\zeta|^2] + \sum_{k=0}^{N-1}\mathbb{E}[u_k^T \int_{t_k}^{t_{k+1}} R(t)dt\; u_k]. \tag{4}$$

2 Algorithm

Setting $\mathbf{x}(t) = \begin{pmatrix} x^T(t) & u^T(t) \end{pmatrix}^T$ we may transform the system (3) in a controlled linear system with finite jumps of the form:

$$d\mathbf{x}(t) = \mathcal{A}_0(t)\mathbf{x}(t)dt + \mathcal{A}_1(t)\mathbf{x}(t)dw(t), \qquad t_k \le t \le t_{k+1} \tag{5a}$$
$$\mathbf{x}(t_k^+) = \mathcal{A}_d\mathbf{x}(t_k) + \mathcal{B}_d u_k, \qquad 0 \le k \le N-1 \tag{5b}$$

and $\mathbf{x}(t_0) = \begin{pmatrix} x_0^T & 0^T \end{pmatrix}^T$ where:

$$\begin{aligned}
\mathcal{A}_k(t) &= \begin{pmatrix} A_k(t) & B_k(t) \\ 0_{mn} & 0_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, k=0,1 \\
\mathcal{A}_d &= \begin{pmatrix} I_n & 0_{nm} \\ 0_{mn} & 0_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, \\
\mathcal{B}_d &= \begin{pmatrix} 0_{nm} \\ I_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times m}.
\end{aligned} \tag{6}$$

The functional (4) becomes

$$\mathcal{J}(\mathbf{x},\zeta,\mathbf{u}) = \mathbb{E}[|z_{\mathbf{u}}(t_f)-\zeta|^2] + \sum_{k=0}^{N-1}\mathbb{E}[u_k^T\mathcal{R}_k u_k] \tag{7}$$

where $\mathcal{R}_k = \int_{t_k}^{t_{k+1}} R(t)dt$ and $z_{\mathbf{u}}(t) = \mathcal{C}(t)\mathbf{x}_{\mathbf{u}}(t)$ with $\mathcal{C}(t) = \begin{pmatrix} C(t) & 0 \end{pmatrix} \in \mathbb{R}^{p\times(n+m)}$ and $\mathbf{x}_{\mathbf{u}}(t)$ is the solution of the initial value problem IVP (5) corresponding to the input $\mathbf{u} = (u_0, u_1, ..., u_{N-1})$.

We have to solve the differential equation with finite jumps on the space $\mathcal{S}_{n+m}$:

$$\dot{X}(t) + \mathcal{A}_0^T(t)X(t) + X(t)\mathcal{A}_0(t) + \mathcal{A}_1^T(t)X(t)\mathcal{A}_1(t) = 0, \tag{8a}$$
$$t_k \le t < t_{k+1}$$
$$X(t_k^-) = \mathcal{A}_d^T X(t_k)\mathcal{A}_d - \mathcal{A}_d^T X(t_k)\mathcal{B}_d(\mathcal{R}_k + \mathcal{B}_d^T X(t_k)\mathcal{B}_d)^\dagger \mathcal{B}_d^T X(t_k)\mathcal{A}_d,$$
$$k = 0,1,...,N-1 \tag{8b}$$
$$X(t_f^-) = \mathcal{C}^T(t_f)\mathcal{C}(t_f). \tag{8c}$$

and of the gain matrices $\tilde{F}_d(k)$ from :

$$\tilde{F}_d(k) \triangleq -(\mathcal{R}_k + \mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)^\dagger\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{A}_d, \quad k=0,1,...,N-1\,, \tag{9}$$

where $\mathcal{R}_k = \int_{t_k}^{t_{k+1}} R(t)dt, \qquad t_k = kh, 0 \le k \le N.$

Here and after, $\mathcal{S}_q \subset \mathbb{R}^{q\times q}$ denotes the linear space of symmetric matrices of size $q \times q$.

<u>**Solve (8a)**</u>.

Consider $t_0 \le t \le t_f$, and $t_0 < t_1 < ... < t_N = t_f$ is a partition of the interval $[t_0, t_f]$, $t_k \le t < t_{k+1}, k = 0, 1, ..., N-1$.

STEP 1.A. We take $X(t_f^-) = \mathcal{C}^T(t_f)\mathcal{C}(t_f),\ (t_f = t_N)$ and compute

$$\tilde{X}(t_{N-1}) = e^{\mathcal{L}^* h}[\mathcal{C}^T(t_f)\mathcal{C}(t_f)], \quad \text{where} \tag{10}$$

$$e^{\mathcal{L}^*h}[\mathbf{X}] \backsimeq \sum_{\ell=0}^{q} \frac{h^\ell}{\ell!}\mathcal{L}^{*\ell}[\mathbf{X}] = \mathbf{X} + h\mathcal{L}^*[\mathbf{X}] + \frac{h^2}{2}(\mathcal{L}^*)[\mathcal{L}^*[\mathbf{X}]] + \ldots + \frac{h^q}{q!}(\mathcal{L}^*)^q[\mathbf{X}]\,,$$

with $q \ge 1$ are sufficiently large.

For the operator $\mathcal{L}^*[X]$ we have

$$\mathcal{L}^*[X] = \mathcal{A}_0^T X + X\mathcal{A}_0 + \mathcal{A}_1^T X \mathcal{A}_1 \tag{11}$$

for all $X = X^T \in \mathbb{R}^{(n+m)\times(n+m)}$.

The iterations $\mathcal{L}^\ell[\mathbf{X}]$ are computed from :

$$\mathcal{L}^{*\ell}[\mathbf{X}] = \mathcal{A}_0^T \mathcal{L}^{*(\ell-1)}[\mathbf{X}] + \mathcal{L}^{*(\ell-1)}[\mathbf{X}]\mathcal{A}_0 + \mathcal{A}_1^T \mathcal{L}^{*(\ell-1)}[\mathbf{X}]\mathcal{A}_1 \tag{12}$$

for $\ell \ge 1$ with $\mathcal{L}^0[\mathbf{X}] = \mathbf{X}$ where $\mathbf{X} = X(t_{j+1}^-)$.

STEP 1.B. We compute the gain matrix $\tilde{F}_d(N-1) \in \mathbb{R}^{m\times(n+m)}$ via (9):

$$\tilde{F}_d(N-1) \triangleq -(\mathcal{R}_{N-1} + \mathcal{B}_d^T \tilde{X}(t_{N-1})\mathcal{B}_d)^\dagger \mathcal{B}_d^T \tilde{X}(t_{N-1})\mathcal{A}_d,$$

STEP 1.C. We compute $\tilde{X}(t_{N-1}^-)$:

$$\tilde{X}(t_{N-1}^-) = \mathcal{A}_d^T \tilde{X}(t_{N-1})\mathcal{A}_d - \mathcal{A}_d^T \tilde{X}(t_{N-1})\mathcal{B}_d(\mathcal{R}_{N-1} + \mathcal{B}_d^T \tilde{X}(t_{N-1})\mathcal{B}_d)^\dagger \mathcal{B}_d^T \tilde{X}(t_{N-1})\mathcal{A}_d,$$

STEP 2. We fix k such that $k \le N-2$. Let us assume that for a $k \le N-2$ were already computed $\tilde{X}(t_{k+1}^-)$.

STEP 2.A. We compute $\tilde{X}(t_k)$ by

$$\tilde{X}(t_k) = e^{\mathcal{L}^* h}[\tilde{X}(t_{k+1}^-)]\,.$$

STEP 2.B. We compute the gain matrix $\tilde{F}_d(k) \in \mathbb{R}^{m\times(n+m)}$ via (9):

$$\tilde{F}_d(k) \triangleq -(\mathcal{R}_k + \mathcal{B}_d^T \tilde{X}(t_k)\mathcal{B}_d)^\dagger \mathcal{B}_d^T \tilde{X}(t_k)\mathcal{A}_d\,.$$

STEP 2.C. We compute $\tilde{X}(t_k^-)$:

$$\tilde{X}(t_k^-) = \mathcal{A}_d^T \tilde{X}(t_k)\mathcal{A}_d - \mathcal{A}_d^T \tilde{X}(t_k)\mathcal{B}_d(\mathcal{R}_k + \mathcal{B}_d^T \tilde{X}(t_k)\mathcal{B}_d)^\dagger \mathcal{B}_d^T \tilde{X}(t_k)\mathcal{A}_d\,.$$

Let $\tilde{\xi}(\cdot)$ be the solution of the following problem with given terminal values

$$\dot{\xi}(t) + \mathcal{A}_0^T(t)\xi(t) = 0, \quad t_k \leq t < t_{k+1} \tag{13a}$$
$$\xi(t_k^-) = (\mathcal{A}_d + \mathcal{B}_d \tilde{F}_d(k))^T \xi(t_k), \quad k = 0, 1, ..., N-1 \tag{13b}$$
$$\tilde{\xi}(t_N^-) = -\mathcal{C}^T(t_f)\zeta \tag{13c}$$

Since (13) is a linear differential equation with finite jumps one obtains that $\tilde{\xi}(t)$ is well defined and right continuous for any $t \in [t_0, t_f]$.

First, one computes $\tilde{\xi}(t_k^-)$ solving the backward equation:

$$\tilde{\xi}(t_k^-) = (\mathcal{A}_d + \mathcal{B}_d \tilde{F}_d(k))^T e^{h\mathcal{A}_0^T} \tilde{\xi}(t_{k+1}^-)$$

with $\tilde{\xi}(t_N^-) = -\mathcal{C}^T\zeta$ where ζ is the desired target. Then one computes

$$\tilde{\xi}(t_k) = e^{h\mathcal{A}_0^T} \tilde{\xi}(t_{k+1}^-), \quad k = N-1, N-2, ..., 1, 0,$$

with

$$e^{h\mathcal{A}_0^T}\xi = \xi + \sum_{k=1}^{p} \frac{h^k}{k!} (\mathcal{A}_0^T)^k \xi$$

with $p \geq 1$ sufficiently large.

One provides the optimal control (see formula (48) in [1])

$$\tilde{u}(t) = \tilde{\mathbb{F}}_d(k)\tilde{x}(t_k) + \tilde{\nu}(k), \quad t_k \leq t < t_{k+1}, 0 \leq k \leq N-1\,,$$

whose coefficients $\mathbb{F}_d(k)$ are obtained from the first n components of $\tilde{F}_d(k)$ (the last m components of $\tilde{F}_d(k)$ are zero).

$\tilde{\nu}(k)$ is computed as (see formula (49) in [1])

$$\tilde{\nu}(k) = -(\mathcal{R}_k + \tilde{X}_{22}(t_k))^\dagger \tilde{\xi}_2(t_k)\,.$$

3 Numerical experiments

We take the matrix coefficients for $t \in [0, 1]\,, n = 4,\ m = 2$:

A0=[1 0 3 0; -4 2 0 -10; -14 8.5 -2.5 0; 0 -2 0 -10];

A1=[0 2 0. -1; 0 0 -3 1.5; -1.45 0.6 -2 0; 0 -3 0 5];

$B0 = [1\ 0; 2\ 5; -1\ 4; 2\ 6];\quad B1 = [0\ 1; -1.5\ -3; 2\ -4; 2\ 0]; \in \mathbb{R}^{4\times 4}$,

$C = [1.0\ -0.25\ -0.75\ -0.5]; \in \mathbb{R}^{1\times n}$,

$R = [0.45\ 0; 0\ 0.75]; \in \mathbb{R}^{2\times 2}$

and $\zeta = 1.5$;

The optimal controls (at the point $t_k = k * h$) $\mathbb{F}_d(k), k = 0, 1, ..., N-1 = 9$ are

$$\mathbb{F}_d(0) = \begin{pmatrix} 0.2958 & 0.5098 & -0.1588 & -0.6806 \\ 0.1392 & -0.1988 & -0.4888 & 0.0924 \end{pmatrix},$$

$$\mathbb{F}_d(1) = \begin{pmatrix} 0.2966 & 0.5071 & -0.1585 & -0.6785 \\ 0.1396 & -0.2002 & -0.4887 & 0.0937 \end{pmatrix},$$

$$\cdots\cdots\cdots$$

$$\mathbb{F}_d(8) = \begin{pmatrix} 0.2416 & 0.4167 & -0.1944 & -0.5877 \\ 0.2010 & -0.2935 & -0.4663 & 0.1718 \end{pmatrix},$$

$$\mathbb{F}_d(9) = \begin{pmatrix} -0.1311 & 0.9683 & -0.2949 & -0.9427 \\ 0.3496 & -0.3983 & -0.4695 & 0.1956 \end{pmatrix}.$$

In order to compute the piecewise constant control $\tilde{u}(t)$we need the vector $\tilde{\nu}(k)$, which can be computed via (49) The values of $\tilde{\nu}(k)$, are

$$\tilde{\nu}(0) = \begin{pmatrix} 0.0009 \\ -0.0019 \end{pmatrix},$$

$$\tilde{\nu}(1) = \begin{pmatrix} 0.0012 \\ -0.0027 \end{pmatrix},$$

$$\cdots\cdots\cdots$$

$$\tilde{\nu}(8) = \begin{pmatrix} -0.0132 \\ 0.0565 \end{pmatrix},$$

$$\tilde{\nu}(9) = \begin{pmatrix} 0.0423 \\ -0.0141 \end{pmatrix},$$

References

[1] V.Dragan, I. Ivanov, *The minimization of the mean square of the deviation of a random signal from a given target*, accepted for publication in Annals of the Academy of Romanian Scientists: Series on Mathematics and its Applications, 2021, vol=13, N 1-2.